

Degradation of non-maximal entanglement of scalar and Dirac fields in non-inertial frames

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Abstract

The entanglement between two modes of the free scalar and Dirac fields as seen by two relatively accelerated observers has been investigated. It is found that the same initial entanglement for an initial state parameter α and its “normalized partner” $\sqrt{1 - \alpha^2}$ will be degraded by the Unruh effect along two different trajectories except for the maximally entangled state, which just shows the inequivalence of the quantization for a free field in the Minkowski and Rindler coordinates. In the infinite acceleration limit the state doesn’t have the distillable entanglement for any α for the scalar field but always remains entangled to a degree which is dependent of α for the Dirac field. It is also interesting to note that in this limit the mutual information equals to just half of the initially mutual information, which is independent of α and the type of field.

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The quantum information theory has made rapid progress in recent years and more and more efforts have been expended on the study of quantum information in the relativistic framework [1]. Especially, the entanglement in a relativistic setting has received considerable attention because it is considered to be a major resource for quantum information tasks such as quantum teleportation, quantum computation and so on [2]. Despite the potential interest to quantum information, the study of entanglement can also help us get a deeper understanding of the black hole thermodynamics [3] and the black hole information paradox [4]. Thus, many authors have investigated the entanglement in the relativistic frames inertial or not for various fields [5, 6, 7].

More recently, Fuentes-Schuller *et al.* [6] and Alsing *et al.* [7] explicitly demonstrated that the entanglement is a quantity depending on a relative acceleration of one of the observers who, before being accelerated, shared a maximally entangled bosonic or fermionic pair. Their results also showed that the different type of field will have a qualitatively different effect on the degradation of entanglement produced by the Unruh effect [8]. Choosing a generic state as the initial entangled state in this Brief Report

$$|\Psi_{sk}\rangle = \sqrt{1-\alpha^2}|0_s\rangle^M|1_k\rangle^M + \alpha|1_s\rangle^M|0_k\rangle^M, \quad (0.1)$$

where α is some real number which satisfies $|\alpha| \in (0, 1)$, α and $\sqrt{1-\alpha^2}$ are the so-called “normalized partners”, we will try to see what effects this uncertainly initial entangled state will on the degradation of entanglement for two relatively accelerated observers due to the presence of an initial state parameter α . Notice that the Schwarzschild space-time very close to the event horizon resembles the Rindler space in the infinite acceleration limit [6, 9]. Hence, as in [6, 7] our results in this limit can be applied to discuss the entanglement between two free bosonic or fermionic modes seen by observers when one observer falls into a black hole and the other barely escapes through eternal uniform acceleration.

Rindler coordinates are appropriate for describing the viewpoint of an observer moving with uniform acceleration. The world lines of uniformly accelerated observers in the Minkowski coordinates correspond to hyperbolae in the left (region I) and right (region II) of the origin which are bounded by light-like asymptotes constituting the Rindler horizon [6, 7], so two Rindler regions are causally disconnected from each other [10]. An observer undergoing uniform acceleration remains constrained to either Rindler region I or II and has no access to the other sector. The system in Eq. (0.1) is bipartite from an inertial perspective, but in a non-inertial frame an extra set of modes in region II becomes relevant. Thus, we will study the mixed-state entanglement of the state as seen by an inertial observer Alice detecting the mode s and a uniformly accelerated observer Bob with proper acceleration a in region I detecting the second mode k .

Bosonic entanglement For a free scalar field, the Minkowski vacuum state can be expressed as a

two-mode squeezed state in the Rindler frame [8, 10]

$$|0_k\rangle^M = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh^n r |n_k\rangle_I |n_k\rangle_{II}, \quad (0.2)$$

where $\cosh r = (1 - e^{-2\pi|k|c/a})^{-1/2}$, k is the wave vector and r is the acceleration parameter, $|n\rangle_I$ and $|n\rangle_{II}$ indicate the Rindler-region-I-particle mode and -II-antiparticle mode respectively. Using Eq. (0.2) and the first excited state [6, 10]

$$|1_k\rangle^M = \frac{1}{\cosh^2 r} \sum_{n=0}^{\infty} \tanh^n r \sqrt{n+1} |(n+1)_k\rangle_I |n_k\rangle_{II},$$

we can rewrite Eq. (0.1) in terms of Minkowski modes for Alice and Rindler modes for Bob. Since Bob is causally disconnected from region II, we will trace over the states in this region and obtain

$$\begin{aligned} \rho_{AB} &= \frac{1}{\cosh^2 r} \sum_{n=0}^{\infty} \tanh^{2n} r \rho_n, \\ \rho_n &= \alpha^2 |1n\rangle\langle 1n| + \frac{\alpha \sqrt{(1-\alpha^2)(n+1)}}{\cosh r} |1n\rangle\langle 0(n+1)| \\ &\quad + \frac{\alpha \sqrt{(1-\alpha^2)(n+1)}}{\cosh r} |0(n+1)\rangle\langle 1n| \\ &\quad + \frac{(1-\alpha^2)(n+1)}{\cosh^2 r} |0(n+1)\rangle\langle 0(n+1)|, \end{aligned} \quad (0.3)$$

where $|nm\rangle = |n_s\rangle^M |m_k\rangle_I$. The partial transpose criterion provides a sufficient condition for the existence of entanglement in this case [11]: if at least one eigenvalue of the partial transpose is negative, the density matrix is entangled; but a state with positive partial transpose can still be entangled. It is well-known bound or nondistillable entanglement [12]. Interchanging Alice's qubits, we get the eigenvalues of the partial transpose $\rho_{AB}^{T_A}$ in the $(n, n+1)$ block

$$\lambda_{\pm}^n = \frac{\tanh^{2n} r}{2 \cosh^2 r} \left[\xi_n \pm \sqrt{\xi_n^2 + \frac{4\alpha^2(1-\alpha^2)}{\cosh^2 r}} \right],$$

where $\xi_n = \alpha^2 \tanh^2 r + (1-\alpha^2)n/\sinh^2 r$. Obviously the eigenvalue λ_-^n is always negative for finite acceleration ($r < \infty$). Hence, this mixed state is always entangled for any finite acceleration of Bob. In the limit $r \rightarrow \infty$, the negative eigenvalue will go to zero. To discuss this further, we will use the logarithmic negativity which serves as an upper bound on the entanglement of distillation [12]. This entanglement monotone is defined as $N(\rho) = \log_2 \|\rho^T\|$, where $\|\rho^T\|$ is the trace norm of the partial transpose ρ^T . We therefore find

$$\begin{aligned} N(\rho_{AB}) &= \log_2 \left\{ \frac{\alpha^2}{\cosh^2 r} + \sum_{n=0}^{\infty} \frac{\tanh^{2n} r}{\cosh^2 r} \right. \\ &\quad \left. \times \sqrt{\left[\alpha^2 \tanh^2 r + \frac{(1-\alpha^2)n}{\sinh^2 r} \right]^2 + \frac{4\alpha^2(1-\alpha^2)}{\cosh^2 r}} \right\}. \end{aligned}$$

For vanishing acceleration ($r = 0$), $N(\rho_{AB}) = \log_2(1 + 2|\alpha|\sqrt{1 - \alpha^2})$. In the range $0 < |\alpha| \leq 1/\sqrt{2}$ the larger α , the stronger the initial entanglement; but in the range $1/\sqrt{2} \leq |\alpha| < 1$ the larger α , the weaker the initial entanglement. For finite acceleration, the monotonous decrease of $N(\rho_{AB})$ with increasing r for different α means that the entanglement of the initial state is lost to the thermal fields generated by the Unruh effect. From Fig. 1 it is surprisingly found that the same initial entanglement for α and its “normalized partner” $\sqrt{1 - \alpha^2}$ will be degraded along two different trajectories except for the maximally entangled state, i.e., $|\alpha| = 1/\sqrt{2}$. This phenomenon, due to the coupling of α and the hyperbolic functions related to r , just shows the inequivalence of the quantization for a scalar field in the Minkowski and Rindler coordinates. The logarithmic negativity is exactly zero for any α in the limit $r \rightarrow \infty$, which indicates that the state doesn't have the distillable entanglement.

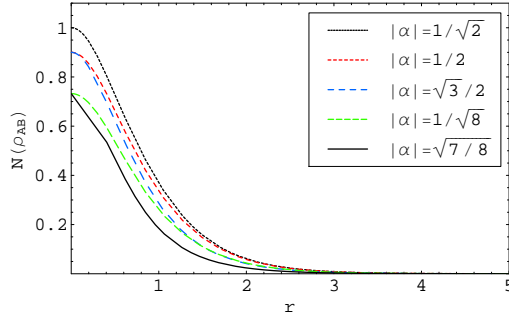


FIG. 1: Logarithmic negativity of the bosonic modes versus r for different α .

The mutual information, which can be used to estimate the total amount of correlations between any two subsystem of the overall system, is defined as [13]

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (0.4)$$

where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the entropy of the density matrix ρ . From Eq. (0.3), we can obtain the entropy of this joint state

$$\begin{aligned} S(\rho_{AB}) &= - \sum_{n=0}^{\infty} \frac{\tanh^{2n} r}{\cosh^2 r} \left[\alpha^2 + \frac{(1 - \alpha^2)(n + 1)}{\cosh^2 r} \right] \\ &\quad \times \log_2 \frac{\tanh^{2n} r}{\cosh^2 r} \left[\alpha^2 + \frac{(1 - \alpha^2)(n + 1)}{\cosh^2 r} \right]. \end{aligned} \quad (0.5)$$

Tracing over Alice's states for ρ_{AB} , we get Bob's density matrix in region I; its entropy is

$$\begin{aligned} S(\rho_{BI}) &= - \sum_{n=0}^{\infty} \frac{\tanh^{2n} r}{\cosh^2 r} \left[\alpha^2 + \frac{(1 - \alpha^2)n}{\sinh^2 r} \right] \\ &\quad \times \log_2 \frac{\tanh^{2n} r}{\cosh^2 r} \left[\alpha^2 + \frac{(1 - \alpha^2)n}{\sinh^2 r} \right]. \end{aligned} \quad (0.6)$$

In the same way, we have Alice's density matrix by tracing over Bob's states; its entropy is given by

$$S(\rho_A) = -[\alpha^2 \log_2 \alpha^2 + (1 - \alpha^2) \log_2 (1 - \alpha^2)]. \quad (0.7)$$

We draw the behaviors of the mutual information $I(\rho_{AB})$ versus r for different α in Fig. 2. For vanishing acceleration, the initially mutual information is $I_{bi} = -2[\alpha^2 \log_2 \alpha^2 + (1 - \alpha^2) \log_2 (1 - \alpha^2)]$. In the range $0 < |\alpha| \leq 1/\sqrt{2}$ the larger α , the stronger I_{bi} ; but in the range $1/\sqrt{2} \leq |\alpha| < 1$ the larger α , the weaker I_{bi} . As the acceleration increases, the mutual information becomes smaller. It is interesting to note that except for the maximally entangled state, the same initially mutual information for α and $\sqrt{1 - \alpha^2}$ will be degraded along two different trajectories. However, in the infinite acceleration limit, the mutual information converges to the same value again, i.e., $I_{bf} = -[\alpha^2 \log_2 \alpha^2 + (1 - \alpha^2) \log_2 (1 - \alpha^2)]$, which equals to just half of I_{bi} . Obviously if I_{bi} is higher, it is degraded to a higher degree in this limit. Since the distillable entanglement in the infinite acceleration limit is zero, we are safe to say that the total correlations consist of classical correlations plus bound entanglement in this limit.

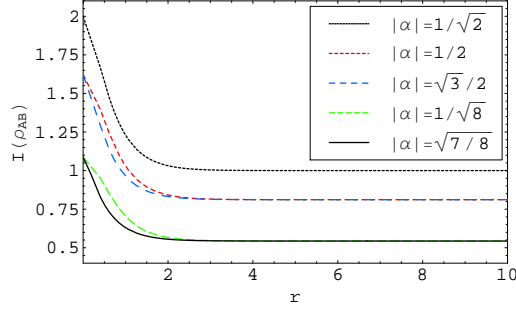


FIG. 2: Mutual information of the bosonic modes versus r for different α .

Fermionic entanglement With the single-mode approximation used by Alsing *et al.*, the fermionic Minkowski vacuum can be written as [7]

$$|0\rangle^M = \cos r |0\rangle_I |0\rangle_{II} + \sin r |1\rangle_I |1\rangle_{II}, \quad (0.8)$$

and the only excited state is given by

$$|1\rangle^M = |1\rangle_I |0\rangle_{II}, \quad (0.9)$$

where $\cos r = (1 + e^{-2\pi\omega c/a})^{-1/2}$ and the acceleration parameter r is in the range $0 \leq r \leq \pi/4$ for $0 \leq a \leq \infty$ in this case. Using Eq. (0.8) and (0.9) for the Minkowski particle states $|0_k\rangle^M$ and $|1_k\rangle^M$ and tracing over the modes in the region II, we get

$$\begin{aligned} \rho_{AB} = & (1 - \alpha^2) |01\rangle\langle 01| \\ & + \alpha \sqrt{(1 - \alpha^2)} \cos r (|01\rangle\langle 10| + |10\rangle\langle 01|) \\ & + \alpha^2 (\cos^2 r |10\rangle\langle 10| + \sin^2 r |11\rangle\langle 11|), \end{aligned} \quad (0.10)$$

with $|mn\rangle = |m\rangle_A^M |n\rangle_{BI}$. The partial transpose criterion provides a necessary and sufficient condition for entanglement in a mixed state of two qubits [11]: if at least one eigenvalue of the partial transpose

is negative, the density matrix is entangled. Interchanging Alice's qubits, we obtain an eigenvalue of the partial transpose $\rho_{AB}^{T_A}$

$$\lambda_- = \frac{1}{2} \left[\alpha^2 \sin^2 r - \sqrt{\alpha^4 \sin^4 r + 4\alpha^2(1 - \alpha^2) \cos^2 r} \right],$$

which is always negative for $0 \leq r \leq \pi/4$. Thus, the state is always entangled for any uniform acceleration of Bob. The logarithmic negativity is expressed as

$$N(\rho_{AB}) = \log_2 [1 - \alpha^2 \sin^2 r + \sqrt{\alpha^4 \sin^4 r + 4\alpha^2(1 - \alpha^2) \cos^2 r}].$$

For vanishing acceleration ($r = 0$), $N(\rho_{AB}) = \log_2(1 + 2|\alpha|\sqrt{1 - \alpha^2})$. For finite acceleration, the

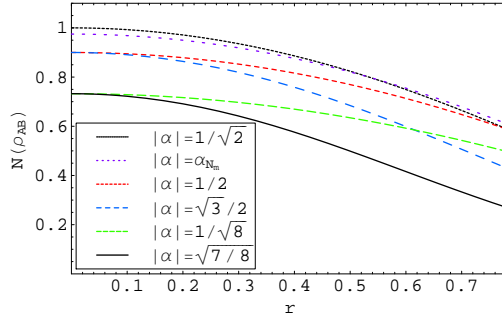


FIG. 3: Logarithmic negativity of the fermionic modes versus r for different α (notice that $\alpha_{N_m} = \sqrt{(4 - \sqrt{2})/7}$).

entanglement is degraded by the Unruh effect just as shown in Fig. 3. We find that in the range $0 < |\alpha| \leq 1/\sqrt{2}$ for the larger α , the initial entanglement is higher, but it isn't always degraded to a higher degree. It should be noted that for $1/2 < |\alpha| < 1/\sqrt{2}$ the final entanglement of the initial state is higher than that of the maximally entangled state, i.e., $\log_2(3/2) \simeq 0.585$, and for $|\alpha| = \sqrt{(4 - \sqrt{2})/7}$ the maximally final entanglement is $\log_2[(5 + 4\sqrt{2})/7] \simeq 0.606$. In the range $1/\sqrt{2} \leq |\alpha| < 1$ the larger α , the weaker the initial entanglement and the lower the final entanglement. Unlike the behaviors of the bosonic case, except for the maximally entangled state, the same initial entanglement of the fermionic modes for α and $\sqrt{1 - \alpha^2}$ will be degraded along two different trajectories and asymptotically reach two differently nonvanishing minimum values in the infinite acceleration limit ($r = \pi/4$) due to the coupling of α and the trigonometric functions related to r . In the infinite acceleration limit $N(\rho_{AB}) = \log_2(1 - \alpha^2/2 + |\alpha|\sqrt{2 - 7\alpha^2/4}) \neq 0$, which means that the state is always entangled. This is in strong contrast to the bosonic case and shows that the fermionic system can be used as a resource for performing certain quantum information processing tasks.

Similar to the bosonic case, we go through the same process again and get the mutual information

for these fermionic modes

$$\begin{aligned}
I(\rho_{AB}) = & (1 - \alpha^2 \sin^2 r) \log_2(1 - \alpha^2 \sin^2 r) \\
& + \alpha^2 \sin^2 r \log_2 \alpha^2 \sin^2 r \\
& - (1 - \alpha^2 \cos^2 r) \log_2(1 - \alpha^2 \cos^2 r) \\
& - \alpha^2 \cos^2 r \log_2 \alpha^2 \cos^2 r \\
& - \alpha^2 \log_2 \alpha^2 - (1 - \alpha^2) \log_2(1 - \alpha^2),
\end{aligned} \tag{0.11}$$

whose trajectories versus r for different α are shown by Fig. 4. For vanishing acceleration, the initially

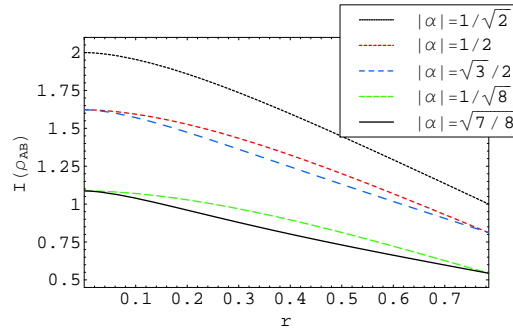


FIG. 4: Mutual information of the fermionic modes versus r for different α .

mutual information is $I_{fi} = -2[\alpha^2 \log_2 \alpha^2 + (1 - \alpha^2) \log_2(1 - \alpha^2)]$, whose behaviors are the same to I_{bi} of the bosonic modes. The mutual information becomes smaller as the acceleration increases, and again we surprisingly find that the same initially mutual information for α and $\sqrt{1 - \alpha^2}$ will be degraded along two different trajectories except for the maximally entangled state. In the infinite acceleration limit the mutual information converges to $I_{ff} = -[\alpha^2 \log_2 \alpha^2 + (1 - \alpha^2) \log_2(1 - \alpha^2)]$, which is just half of I_{fi} . This behavior is reminiscent of that seen for the bosonic case, so we conclude that

$$I_f = \frac{1}{2} I_i, \tag{0.12}$$

which is independent of α and the type of field.

It should be noted that if we set the initial entangled state as

$$|\Psi_{sk}\rangle = \alpha|0_s\rangle^M|0_k\rangle^M + \sqrt{1 - \alpha^2}|1_s\rangle^M|1_k\rangle^M, \tag{0.13}$$

we will have the same behavior of the entanglement degradation for the same α just as shown in Figs. 1-4.

Summarizing, the entanglement of the scalar and Dirac fields in non-inertial frames is degraded by the Unruh effect as the Bob's rate of acceleration increases, but their behaviors of the degradation of entanglement are different for the same initial state parameter α . It is surprisingly found that

the same initial entanglement for α and $\sqrt{1-\alpha^2}$ will be degraded along two different trajectories except for the maximally entangled state, which just shows the inequivalence of the quantization for a free field in the Minkowski and Rindler coordinates. In the infinite acceleration limit, which can be applied to the case Alice falling into a black hole while Bob barely escapes, the state doesn't have the distillable entanglement for any α for the scalar field but always remains entangled to a degree which is dependent of α for the Dirac field. It should be noted that for $|\alpha| = \sqrt{(4-\sqrt{2})/7}$, we will have the maximally final entanglement for the fermionic state in this limit. Further analysis shows that the mutual information is degraded to a nonvanishing minimum value which is dependent of α for these two fields with increasing acceleration parameter r . However, it is interesting to note that the mutual information in the infinite acceleration limit equals to just half of the initially mutual information, which is independent of α and the type of field.

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